Using the standard recursion relation,

$$(n-m)P_n^m = (2n-1)P_{n-1}^m - (n+m-1)P_{n-2}^m$$

and the fact that $P_n^m = 0$ for $m > n$, a simple relation can be obtained for P_{n+1}^n:

$$P_{n+1}^n = (2n+1)P_n^n \tag{5}$$

Then, Eqs. (4) and (5) can be used to compute any P_n^m given the initial value $P_1^1 = (1-x^2)^{1/2}$.

Alternative 2

An alternative mechanization of Eq. (3) is

$$P_n^{m-1} = \frac{m\alpha P_n^m - P_n^{m+1}}{(n+m)(n-m+1)} \tag{6}$$

In this case, we first compute all P_n^n up to $n = N$ using Eq. (4) with the starting value $P_1^1 = (1-x^2)^{1/2}$. Equation (6) is then solved starting with P_n^n and proceeding with decreasing m to $m = 1$ for given n. Where appropriate, we again make use of the relationship $P_n^m = 0$ for $m > n$ in utilizing Eq. (6). In contrast to Eq. (3) applied in the forward sense, Eq. (6) is stable when run backward over m. An analogous instability has been discussed for Bessel functions.[4]

Conclusions

Use of the standard recursion relationships for the associated Legendre functions will, in general, lead to numerical roundoff errors in the computed gravity. These roundoff errors can avalanche catastrophically near the poles unless the recursion relations used are stable. Two examples of stable recursion relations have been discussed and an unstable one analyzed.

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The Minimum for Geometric Dilution of Precision in Global Positioning System Navigation

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IN a previous Note,[1] the author presented some simple bounds to the global positioning system (GPS) navigation performance index, the geometric dilution of precision (GDOP). It was shown that the GDOP must be greater than $\sqrt{2}$ and that a value as low as $\sqrt{2.5}$ was attained for a completely symmetrical GPS configuration; i.e., the line-of-sight vectors from the user to the four GPS satellites were all separated by the same angle $\cos^{-1}(-1/3)$. It will be shown below that the value of $\sqrt{2.5}$ is indeed the minimum for GDOP.

As shown in Ref. 1,

$$\text{GDOP} = \sqrt{1/\lambda_1 + 1/\lambda_2 + 1/\lambda_3 + 1/\lambda_4} \tag{1}$$

where the λ are eigenvalues of a 4×4 real symmetric nonnegative matrix with a trace equal to 8. The results of Ref. 1 were obtained by examining the 4×4 matrix HH^T, but it was pointed out that this 4×4 matrix may also be

$$H^TH = \begin{bmatrix} aa^T + bb^T + cc^T + dd^T & a+b+c+d \\ (a+b+c+d)^T & 4 \end{bmatrix} \tag{2}$$

where

$$H^T = \begin{bmatrix} a & b & c & d \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is the measurement partial derivative matrix, transposed; *a, b, c,* and *d* are line-of-sight unit vectors from a set of four GPS satellites to the user. From the well-known mini-max property of the eigenvalues of symmetric matrices,[2] one has, for the largest eigenvalue λ_4 of H^TH,

$$\lambda_4 \geq [0\ 0\ 0\ 1]\,(H^TH)\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 4 \tag{3}$$

Subject to this constraint and the fact that all the λ are nonnegative and have a sum equal to 8, one has

$$\text{GDOP} \geq \text{minimum}\sqrt{[1/\lambda_4 + 3/[(8-\lambda_4)/3]} = \sqrt{2.5}$$

occurring at $\lambda_4 = 4$. That 2.5 is a minimum and not a lower bound has already been shown by the construction of the completely symmetrical GPS configuration.

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Optimization of Cruise at Constant Altitude

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Nomenclature

C = cost function
C_{D_0} = drag coefficient at zero lift

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